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THEORY OF OPTIMAL PROCESSES

-USSR-

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In recent times, in the theory of automatic control, special attention is paid to ensure very fast control, which led to the appearance of a number of works devoted to the study of the so-called optimal processes (cf (1), where one may find the bibliography of the subject). Here we give a general approach to the study of optimal processes.

1. Formulation of the problem. Let us consider a representation of a point $x = (x^1, \ldots, x^n)$ in an n-dimensional phase space, whose equations of motion are stated in the usual manner

$$\dot{x}^i = f^i(x^1, \ldots, x^n; u^1, \ldots, u^r) = f^i(x, u), \quad i = 1, \ldots, n. \qquad (1)$$

Here, $u^1, \ldots, u^r$ are control parameters. If the control mode is known, i.e., a variable vector $u(t) = (u^1(t), \ldots, u^r(t))$ is known in an r-dimensional space, then the system (1) uniquely describes the motion of the point.

We impose the natural conditions of piecewise continuity and piecewise smoothness of the vector $u(t)$, and therefore assume that the variable vector $u(t)$ is in a constant closed domain $\bar{\Omega}$ of the space of variables $u^1, \ldots, u^r$, which is called the closure of the open domain Ω with piecewise smooth $(r - 1)$-dimensional boundary. For example, the domain may be an r-dimensional cube such that $|u^i| \leq 1$, $i = 1, \ldots, r$, a half space $u^1 \geq 0$, etc. The control vector $u(t)$, satisfying the stated conditions shall be called an admissible one.

Statement of the general problem. In the phase space $x^1, \ldots, x^n$, there exist two points ξ_0, ξ_1. An admissible controlling vector $u(t)$ is to be chosen in such a way that the point from position ξ_0 should arrive at position ξ_1 after a minimum of time.

The desired control vector $u(t)$ shall be called the optimal control, the corresponding trajectory $x(t) = (x^1(t), \ldots, x^n(t))$ of

system (1) is called the optimal trajectory.

2. <u>The necessary conditions for optimality</u>. Let us assume that there exist the optimal directions u(t) and corresponding to it, the optimal trajectory x(t). The trajectory x(t) satisfies the boundary conditions $x(t_o) = \xi_o$, $x(t_1) = \xi_1$. Let us assume initially that the directing vector u(t) for $t_o \leq t \leq t_1$ is properly contained in the open domain Ω. It follows that for arbitrary perturbations of sufficiently small modulus $\delta u(t) = (\delta u^1(t), \ldots, \delta u^r(t))$ of the vector u(t), the direction $u(t) + \delta u(t)$ shall be in the domain Ω. We shall denote $x + \delta x$ the "perturbed" (i.e., corresponding to the direction $u(t) + \delta u(t)$), the trajectory of a point with a previously stated initial condition $x(t_o) + \delta x(t_o) = \xi_o$ i.e., $\delta x(t_o) = 0$. The linear approximation equations $\delta_I x = (\delta_I x^1, \ldots, \delta_I x^n)$ for the perturbations $\delta x = (\delta x^1, \ldots, \delta x^n)$ have the form

$$\delta_1 \dot{x}^i = \frac{\partial f^i}{\partial x^\alpha} \delta_1 x^\alpha + \frac{\partial f^i}{\partial u^\beta} \delta u^\beta; \quad \delta_1 x(t_0) = 0; \quad i = 1, \ldots, n. \tag{2}$$

As the consequence of the linearity of system (2), the points $x(t_1) + \delta_I x(t_1)$ corresponding to all, for a sufficiently small modulus, perturbations $\delta_I u(t)$ fill the domain of some linear manifold P' which passes through the point $x(t_1)$. It follows easily from the optimality of the trajectory x(t), that the dimensionality of the manifold P' is at most n - 1, and P', generally speaking, is not tangent to the trajectory x(t). Let $P(t_1)$ be some (n - 1)-dimensional plane which contains P' and which is not tangent to the trajectory x(t). The covariant coordinates of the (n - 1)-dimensional plane $P(t_1)$ are denoted by $a_1, \ldots, a_n$, and then $a_\alpha \delta_I x^\alpha(t_1) = 0$.

Assume that $\varphi_j(t) = (\varphi_j^1(t), \ldots, \varphi_j^n(t))$, j = 1, ..., n is the fundamental system of solutions of the homogeneous system corresponding to system (2), and $\|\psi_j^i(t)\|$ is a matrix which is the inverse of the $\|\varphi_j^i(t)$ matrix. The solution of system (2) may be expressed by

$$\delta_1 x^i(t) = \varphi_\alpha^i(t) \int_{t_0}^{t} \psi_\beta^\alpha \frac{\partial f^\beta}{\partial u^\gamma} \delta u^\gamma d\tau, \quad i = 1, \ldots, n. \tag{3}$$

Using the equality $a_\alpha \delta_1 x^\alpha(t_1) = 0$, we have

$$a_\alpha \delta_1 x^\alpha(t_1) = a_\alpha \varphi_\beta^\alpha(t_1) \int_{t_0}^{t_1} \psi_\gamma^\beta \frac{\partial f^\gamma}{\partial u^\nu} \delta u^\nu d\tau = 0.$$

Let us denote $a_\alpha \varphi_\beta^\alpha(t_1) = b_\beta, \; b_\beta \psi_\gamma^\beta(t) = \psi_\gamma(t).$

then $a_\alpha \delta_1 x^\alpha(t_1) = \int_{t_0}^{t_1} \psi_\alpha \frac{\partial f^\alpha}{\partial u^\beta} \delta u^\beta d\tau = 0.$

Since $\delta u(t) = (\delta u^1(t), \ldots, \delta u^r(t))$ is an arbitrary, of sufficiently small modulus, perturbation, it follows from the last equation that the system of equations is $\psi_\alpha(t) \frac{\partial f^\alpha}{\partial u^i} = 0, \quad t_0 \leq t \leq t_1, \quad i = 1, \ldots, r.$ (4)

The vector $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ has a simple geometrical interpretation. The point $x(t) + \delta_I x(t)$ lies in the (n - 1)-dimensional plane P(t), in which lies the point x(t) and which has the

covariant coordinate system: $\psi_1(t), \ldots, \psi_n(t)$. In particular $(\psi_1(t_1), \ldots, \psi_n(t_1)) = (a_1, \ldots, a_n)$. Using the function $\psi_i(t) = b_\alpha \psi_i^\alpha(t)$, $i = 1, \ldots, n$, we obtain the system of differential equations for $\psi_i(t)$:

$$\dot{\psi}_i(t) = -\frac{\partial f^\alpha}{\partial x^i}\psi_\alpha, \quad i = 1, \ldots, n. \qquad (5)$$

Combining the systems (1), (4) and (5), we have

$$\left.\begin{aligned} &\dot{x}^i = f^i(x, u), \quad i = 1, \ldots, n; \\ &\dot{\psi}_i = -\frac{\partial f^\alpha}{\partial x^i}\psi_\alpha, \quad i = 1, \ldots, n; \\ &\psi_\alpha \frac{\partial f^\alpha}{\partial u^j} = 0, \quad t_0 \leqslant t \leqslant t_1, \quad j = 1, \ldots, r. \end{aligned}\right| \qquad (6)$$

The system (6) represents the totality of the necessary conditions which the optimal direction u(t) must satisfy. u(t) is properly contained in the open domain Ω and with it are associated the optimal trajectory x(t) and the vector $\psi(t)$.

Multiplying the vector $\psi(t)$ by a suitable constant (which does not change the trajectory x(t) nor the direction u(t)), we may obtain the following condition: $\psi_\alpha(t_0) f^\alpha(x(t_0)), u(t_0)) > 0$. As the plane P(t) is not the tangent plane to the trajectory x(t), i.e. $\psi_\alpha f^\alpha \neq 0$ for any t, then at any time the inequality $\psi_\alpha f^\alpha > 0$ shall be satisfied.

Now, if one should assume that the optimal direction is in the closed domain $\bar{\Omega}$ and we consider the inequality $\psi_\alpha f^\alpha |_{t = t_0} > 0$ then the system (4) of the necessary conditions shall become a more general condition as below

$$\psi_\alpha \frac{\partial f^\alpha}{\partial u^\beta}\delta u^\beta \leqslant 0, \quad t_0 \leqslant t \leqslant t_1, \qquad (7)$$

for arbitrary perturbations $\delta u^\beta(t)$, on which we have "natural constraints", which follow from the condition that $u(t) + \delta u(t) \in \bar{\Omega}$.

3. <u>The sufficient conditions of optimality (locally)</u>. At this point we again assume that the direction vector u(t) is properly contained in the domain Ω and satisfies the necessary conditions (6). The equations of the second approximation $\delta_{II}x$ for the perturbation δx have the form

$$\delta_{II}\dot{x}^i = \frac{\partial f^i}{\partial x^\alpha}\delta_{II}x^\alpha + \frac{\partial f^i}{\partial u^\beta}\delta u^\beta + B^i(t),$$

$$B^i(t) = \frac{1}{2}\left[\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}\delta_I x^\alpha \delta_I x^\beta + 2\frac{\partial^2 f^i}{\partial x^\alpha \partial u^\beta}\delta_I x^\alpha \delta u^\beta + \frac{\partial^2 f^i}{\partial u^\alpha \partial u^\beta}\delta u^\alpha \delta u^\beta\right].$$

The point whose coordinates are $x^i(t) + \delta_{II}x^i(t) = x^i(t) + \delta_I x^i(t) + \varphi^i_\alpha(t)\int_{t_0}^{t}\psi^\alpha_\beta B^\beta d\tau$

no longer lies in the plane P(t). If the moving point has passed the plane P(t) when the motion was perturbed at time t, then the scalar product is positive,

$$\psi_\alpha(t)\,\delta_{II}x^\alpha(t) = \psi_\alpha(t)\,\delta_I x^\alpha(t) + \psi_\alpha(t)\,\varphi^\alpha_\beta(t)\int_{t_0}^{t}\varphi^\beta_\gamma B^\gamma d\tau =$$

$$\psi_\alpha(t)\,\varphi^\alpha_\beta(t)\int_{t_0}^{t}\varphi^\beta_\gamma B^\gamma d\tau = \int_{t_0}^{t}\psi_\alpha B^\alpha d\tau$$

Howver, if the point has not yet reached the plane $P(t)$, then $\psi_\alpha(t)\delta_{II}x^\alpha(t) = \int_{t_0}^{t}\psi_\alpha B^\alpha d\tau < 0.$ The bilinear form $\psi_\alpha \frac{\partial^2 f^\alpha}{\partial u^i \partial u^k}\delta u^i \delta u^k$ (of the variables $u^1, \ldots, u^r$), at the point $(x(t_0), u(t_0), t_0)$, is negative definite. Then the scalar product is

$$\psi_\alpha(t)\,\delta_{II}x^\alpha(t) = \int_{t_0}^{t}\psi_\alpha B^\alpha d\tau < 0$$

for arbitrary, sufficiently small modules of perturbations $\delta u(t)$ and sufficiently small difference $t - t_0$. In this case the direction $u(t)$ and the trajectory $x(t)$ are locally optimal, i.e., the point $x(t_0)$ may be contained in such a small neighborhood V, such that if $x(t')$ and $x(t'')$, (for $t' < t''$), are two arbitrary points on the trajectory belonging to V, then for no direction, sufficiently close to $u(t)$, one may reach the point $x(t'')$ from the point $x(t')$ during time which is less than $t'' - t'$.

If the form $\psi_\alpha \frac{\partial^2 f^\alpha}{\partial u^i \partial u^k}\delta u^i \delta u^k$ at the point $(x(t_0), u(t_0), t_0)$ is indefinite, then (for some sufficiently general additional conditions) no direction $u(t)$ being close to the time $t = t_0$, being properly contained in the domain Ω, may be optimal, even locally. If, however, there exist optimal trajectories through the point $x(t_0)$, then the corresponding direction vectors $u(t)$ in the neighborhood of $t = t_0$ should lie on the boundary of the closed domain Ω.

4. <u>The Maximum Principle</u>. From system (6) and the fact that the bilinear form $\psi_\alpha \frac{\partial^2 f^\alpha}{\partial u^i \partial u^k}\delta u^i \delta u^k$ is negative definite, we have that the expression $\psi_\alpha(t) f^\alpha(x(t), u(t))$ reaches the respective maximum for constant vectors $x(t)$, (t) and the variable vector $u(t)$; for sufficiently small (with respect to modulus) perturbations $\delta u(t)$ we have this inequality

$$\psi_\alpha(t) f^\alpha(x(t), u(t)) \geqslant \psi_\alpha(t) f^\alpha(x(t), u(t) + \delta u(t))$$

for all times, provided that the equations (6) are satisfied and the bilinear form is negative definite.

The above is a special case of the discussed general principle, the principle which we call the Maximum Principle (this principle has been proved by us only for some special cases up till now):

Assume that the function $H(x, \psi, u) = \psi_\alpha f^\alpha(x\ u)$ has a maximum with respect to u for arbitrary constant x and ψ, provided that the vector u varies in the closed domain Ω. This maximum we denote by $M(x, \psi)$. If the 2n-dimensional vector (x, ψ) is a solution of the hamiltonian system

$$\left.\begin{aligned} \dot{x}^i &= f^i(x, u) = \frac{\partial H}{\partial \psi_i}, \\ \dot{\psi}_i &= -\frac{\partial f^a}{\partial x^i}\psi_a = -\frac{\partial H}{\partial x^i}, \end{aligned}\right\} \quad i = 1, \ldots, n, \tag{8}$$

where the piecewise continuous vector $u(t)$ satisfies the condition $H(x(t), \psi(t), u(t)) = M(x(t), \psi(t)) > 0$ for all t, then $u(t)$ is defined to be the optimal direction and $x(t)$ the corresponding optimal (locally) trajectory of system (1).

We shall assume a constant initial condition $x(t_0) = \xi_0$ and as much as possible shall endeavor to specify the initial condition $\psi(t_0) = \eta_0$. Then, the system (8) together with these initial conditions and the condition $H(x(t), \psi(t), u(t)) = M(x(t), \psi(t)) > 0$ define the set of all optimal (locally) trajectories passing through the point $x(t_0) = \xi_0$, and optimal directions $u(t)$ corresponding to these trajectories.

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